

Applications of hidden symmetries to black hole physics

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Abstract. This work is a brief review of applications of hidden symmetries to black hole physics. Symmetry is one of the most important concepts of the science. In physics and mathematics the symmetry allows one to simplify a problem, and often to make it solvable. According to the Noether theorem symmetries are responsible for conservation laws. Besides evident (explicit) spacetime symmetries, responsible for conservation of energy, momentum, and angular momentum of a system, there also exist what is called hidden symmetries, which are connected with higher order in momentum integrals of motion. A remarkable fact is that black holes in four and higher dimensions always possess a set ('tower') of explicit and hidden symmetries which make the equations of motion of particles and light completely integrable. The paper gives a general review of the recently obtained results. The main focus is on understanding why at all black holes have something (symmetry) to hide.

1. Introduction

In this paper we discuss higher dimensional rotating black holes and their properties. We consider black holes with the spherical topology of the horizon in a spacetime which is either asymptotically flat (vacuum) or (A)deSitter (with cosmological constant). We demonstrate that such solutions of the Einstein equations always possess a "tower" of symmetries, which make the equations of geodesic motion completely integrable. This 'tower' is generated by a single object, which we call a *principal conformal Killing-Yano tensor*. This object is responsible for a set of Killing vectors reflecting a spacetime symmetry. It also generates a set of Killing tensors, connected with hidden symmetries of the spacetime. The corresponding conserved quantities, which are first and second order in the momentum, form a complete set of integrals of motion which make the geodesic equation completely integrable. The purpose of the paper is to describe a general scheme of this construction, and to discuss its application to concrete problems connected with particle motion and field propagation in the higher dimensional black holes. We also describe a class of metrics which admit the principal conformal Killing-Yano tensor and its generalizations.

2. Complete integrability

2.1. Phase space

Let us first remind three related notions which play an important role in study of dynamical systems: (i) complete integrability, (ii) separation of variables, and (iii) hidden symmetries (for more details see [1, 2]).

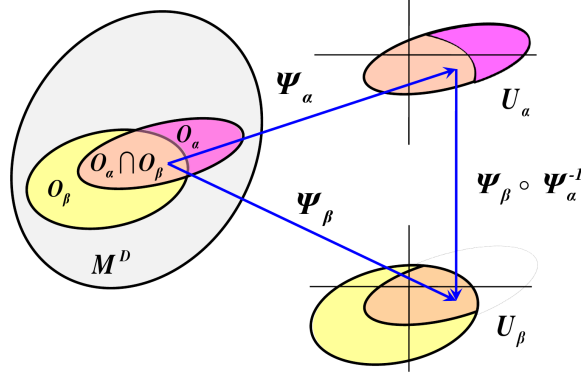


Figure 1. Coordinates on n -dimensional manifold are maps Ψ_α of open regions O_α on the regions U_α in the n -dimensional Euclidean space R^n . The map $\Psi_\beta \circ \Psi_\alpha^{-1}$ determines a transformation between two coordinate systems.

A *phase space* is a set of three items $\{M^{2m}, \Omega, H\}$. M^{2m} is a $2m$ -dimensional differential manifold. A symplectic form Ω is a closed, $d\Omega = 0$, non-degenerate 2-form. The non-degeneracy means that the corresponding matrix of its coefficient has the rank $2m$. Locally $\Omega = 0$ can be presented in the form

$$\Omega = d\alpha, \quad (1)$$

where α is a 1-form. H is a scalar function on M^{2m} called a *Hamiltonian*. We denote coordinates on M^{2m} by z^A ($A = 1, \dots, 2m$). A set of coordinates which covers all the manifold is called an *atlas*. Figure 1 illustrates these definitions.

Since the symplectic form is non-degenerate, the tensor Ω_{AB} has an inverse one Ω^{AB} , defined by the relation $\Omega^{AB}\Omega_{BC} = \delta_C^A$. The tensors Ω_{AB} and Ω^{AB} can be used to relate tensors with upper and lower position of indices. For example,

$$\eta^A = \Omega^{AB}H_{,B} \quad (2)$$

is a vector generating the Hamiltonian flow. The equation of motion of a particle in the phase space is

$$\dot{z}^A = \eta^A. \quad (3)$$

Poisson bracket for two functions on the phase space A and B is defined as

$$\{A, B\} = \Omega^{AB}A_{,A}B_{,B}. \quad (4)$$

One says that these function are in *involution* if their Poisson bracket vanishes. The Poisson brackets for any three functions A , B , and C on the symplectic manifold obey the *Jacobi identity*

$$\{\{A, B\}, C\} + \{\{B, C\}, A\} + \{\{C, A\}, B\} = 0. \quad (5)$$

If $F(z)$ is a function on the phase space, the equation of motion Eq.(3) determines its evolution $F_t = F(z(t))$

$$\dot{F}_t = \{H, F_t\}. \quad (6)$$

A function F for which $\{H, F\} = 0$ is an integral of motion of the Hamiltonian system. The Hamiltonian itself is a trivial integral of motion.

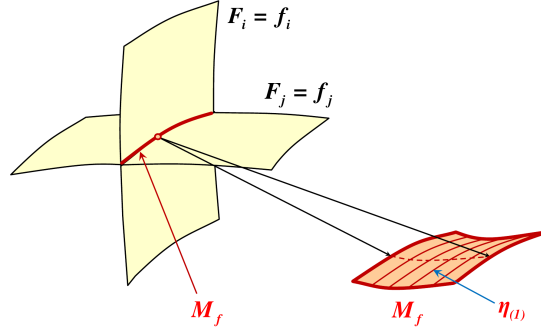


Figure 2. Illustration to the proof of Liouville theorem.

According to *Darboux theorem*, in the vicinity of any point of the phase space it is always possible to choose *canonical coordinates* $z^A = (q_1, q_2, \dots, q_m, p_1, p_2, \dots, p_m)$ in which the symplectic form is

$$\Omega = \sum_{i=1}^m dp_i \wedge dq_i. \quad (7)$$

In such canonical coordinates many relations are simplified and take a well known form. A set of canonical coordinates which covers the phase space is called a *canonical atlas*.

2.2. Integrability

Integrability of equations of a dynamical system generically means that these equations can be solved by quadratures. Integrability is linked to ‘existence of constants of motion’. For the integrability it is important to know: (i) How many constants of motion exist? (ii) How precisely are they related? and (iii) How the phase space is foliated by their level sets?

A system of differential equations is said to be integrable by quadratures if its solution can be found after a finite number of steps involving algebraic operations and integration. The following theorem (Bour, 1855; Liouville, 1855) establishes conditions required for the complete integrability of a Hamiltonian system:

If a Hamiltonian system with m degrees of freedom has m integrals of motion $F_1 = H, F_2, \dots, F_m$ in involution which are functionally independent on the intersection sets of the m functions, $F_i = f_i$, the solutions of the corresponding Hamiltonian equations can be found by quadratures.

The main idea behind Liouville’s theorem is that the first integrals of motion F_i can be used as m coordinates. The involution condition implies that the m vector fields generated by gradients of F_i commute with each other and provide a choice of canonical coordinates. In these coordinates, the Hamiltonian is effectively reduced to a sum of m decoupled Hamiltonians that can be integrated.

We denote by M_f an intersection of the level sets $F_i = f_i$ for m integrals of motion (see Figure 2). Since the integrals of motion are independent, the tangent to M_f vectors $X_i^A = \Omega^{AB} F_{i,B}$ are linearly independent and M_f is a m –dimensional submanifold of the phase space. The condition that the integrals of motion are in involution implies

$$[X_i, X_j] = 0. \quad (8)$$

One also has

$$\Omega_{AB}X_i^AX_j^B = \{F_i, F_j\} = 0. \quad (9)$$

Let us use the equations $F_i(p, q) = f_i$ to get $p_i = p_i(f, q)$. Denote

$$S(F, q) = \int_{q^0}^q \sum_{i=1}^m p_i(f, q) dq_i. \quad (10)$$

The integral is taken over a path on M_f connecting two its points q^0 and q . The relation Eq.(9) implies that $S(F, q)$ does not depend on the choice of the path. Denote

$$\Psi_i = \frac{\partial S}{\partial F_i}, \quad (11)$$

then

$$dS = \sum_{i=1}^m (\Psi_i dF_i + p_i dq_i). \quad (12)$$

Since the symplectic form Ω is closed, one has $d^2S = 0$ and hence

$$\Omega = \sum_{i=1}^m dp_i \wedge dq_i = \sum_{i=1}^m d\Psi_i \wedge dF_i. \quad (13)$$

This result means that there exists a canonical transformation (p_i, q_i) to (F_i, Ψ_i) . To obtain ‘new’ canonical coordinates two operations are required: (1) finding of $p_i = p_i(f, q)$, and (2) calculation of some integrals. In the new variables the Hamiltonian equations take the form

$$\dot{F}_i = \{H, F_i\} = 0, \quad (14)$$

$$\dot{\Psi}_i = \{H, \Psi_i\} = \frac{\partial H}{\partial F_i} = \delta_i^1. \quad (15)$$

The solution of this system is trivial

$$F_i = \text{const}, \quad \Psi_i = a_i + \delta_i^1 t. \quad (16)$$

Complete integrability and chaotic motion are at the two ends of ‘properties’ of a dynamical system. The integrability is exceptional, while the chaoticity is generic. In all cases, integrability seems to be deeply related to some symmetry, which might be partially hidden. The existence of integrals of motion reflects the symmetry.

Important known examples of completely integrable mechanical systems include:

- (i) Motion in Euclidean space under a central potential;
- (ii) Motion in the two Newtonian fixed centers;
- (iii) Geodesics on an ellipsoid;
- (iv) Motion of a rigid body about a fixed point;
- (v) Neumann model¹.

¹ The Lagrangian of the Neumann model is

$$L = \frac{1}{2} \sum_{k=1}^N [\dot{x}_k^2 - a_k x_k^2 + \Lambda(x_k^2 - 1)] . \quad (17)$$

3. Separation of variables and integrability

Complete integrability of the Hamiltonian systems is closely related with the separation of variables in the Hamilton-Jacobi equation. For a Hamiltonian $H(P, Q)$, $P = p_1, \dots, p_m$ and $Q = q_1, \dots, q_m$, the *Hamilton-Jacobi equation* is

$$H(\partial_Q S, Q) = 0, \quad \partial_Q S = (\partial_{q_1} S, \dots, \partial_{q_m} S). \quad (18)$$

If a variable q_1 and a derivative $\partial_{q_1} S$ enter this equation in a form of single expression $\Phi_1(\partial_{q_1} S, q_1)$, then one says that the variable q_1 is separated. In such a case one may try to search a solution in the form

$$S = S_1(q_1) + S'(q_2, \dots, q_m). \quad (19)$$

Putting

$$\Phi_1(\partial_{q_1} S, q_1) = C_1, \quad (20)$$

one obtains an equation with a less number of variables

$$H_1(\partial_{q_2} S', \dots, \partial_{q_m} S', q_2, \dots, q_m; C_1) = 0. \quad (21)$$

Let $S'(q_2, \dots, q_m; C_1)$ be a solution of this equation depending on a parameter C_1 , then Eq.(19) is a solution of Eq.(18) when S_1 obeys an ordinary differential equation Eq.(20), which is easily solvable by quadratures. If a variable q_2 can be separated in a new equation Eq.(21) one can repeat this procedure again. One says that the Hamilton-Jacobi equation Eq.(18) allows a *complete separation of variables* if after m steps we obtain a solution of the initial equation Eq.(18) which contains m constants C_i

$$S = S_1(q_1, C_1) + S_2(q_2; C_1, C_2) + \dots + S_m(q_m; C_1, \dots, C_m). \quad (22)$$

In this case one obtains a complete solution of the Hamilton-Jacobi equation which depends on m parameters and the corresponding Hamilton equations are integrable by quadratures (*Jacobi theorem*).

The constants C_1, \dots, C_m for a completely separable Hamilton-Jacobi equation can be considered as functions on the phase space where they are integrals of motion. In a case, when these integrals on motion are independent and in involution, the system is completely integrable in the sense of Liouville.

4. Particle motion in General Relativity

4.1. Equation of motion in the Hamiltonian form

Consider a particle motion of mass m in the gravitational field. Its equation of motion is

$$m \frac{D^2 x^\mu}{d\tau^2} = 0. \quad (23)$$

Here, $D/d\tau$ is the covariant derivative with respect to the proper time τ . Introduce the affine parameter $\lambda = \tau/\mu$ and denote a derivative with respect to it by a dot. The Lagrangian for Eq.(23) is

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu. \quad (24)$$

The momentum p_μ is defined as follows

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = g_{\mu\nu} \dot{x}^\nu. \quad (25)$$

Denote by D the dimensionality of the spacetime. Coordinates (p_μ, x^μ) are canonical on the phase space M^{2D} . In these coordinates the symplectic form is

$$\Omega = \sum_{\mu=1}^D dp_\mu \wedge dx^\mu. \quad (26)$$

The Hamiltonian of the particle is

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu. \quad (27)$$

It gives trivial integral of motion

$$H = \frac{1}{2} m^2. \quad (28)$$

For null rays one must put $m = 0$. The Hamiltonian equations of motion

$$\dot{x}^\mu = \{H, x^\mu\} = g^{\mu\nu} p_\nu, \quad (29)$$

$$\dot{p}_\mu = \{H, p_\mu\} = -\frac{1}{2} g^{\nu\lambda}_{,\mu} p_\nu p_\lambda, \quad (30)$$

are equivalent to the geodesic equation Eq.(23), which can be written in the form

$$p^\nu p_{\mu;\nu} = 0. \quad (31)$$

4.2. Integrals of motion and Killing tensors

Consider a special monomial in the momentum on the phase space of the relativistic particle of the form

$$\mathcal{K} = K^{\mu_1 \dots \mu_s} p_{\mu_1} \dots p_{\mu_s}. \quad (32)$$

A condition that this is an integral of motion implies

$$K^{(\mu_1 \dots \mu_s; \nu)} = 0. \quad (33)$$

The symmetric tensor of the rank s , $K_{\mu_1 \dots \mu_s}$, which obeys the equation Eq.(33), is called a *Killing tensor*. The metric $g_{\mu\nu}$ is a trivial example of the Killing tensor of rank 2.

Suppose we have two integrals of motion, $\mathcal{K}_{(s)}$ and $\mathcal{K}_{(t)}$, connected with the Killing tensors $K_{(s)}^{\mu_1 \dots \mu_s}$ and $K_{(t)}^{\nu_1 \dots \nu_t}$, respectively. Using the *Jacobi identity* Eq.(5) it is easy to check that $\{\mathcal{K}_{(s)}, \mathcal{K}_{(t)}\}$ also commutes with the Hamiltonian H and hence is an integral of motion. This commutator is a monomial of the power $s + t - 1$ in the momentum. The corresponding Killing tensor of the rank $s + t - 1$ is

$$[K_{(s)}, K_{(t)}] = K_{(s+t-1)}, \quad (34)$$

$$K_{(s+t-1)}^{\mu_1 \dots \mu_{s-1} \lambda \nu_1 \dots \nu_{t-1}} = s K_{(s)}^{\epsilon(\mu_1 \dots \mu_{s-1}} \partial_\epsilon K_{(t)}^{\lambda \nu_1 \dots \nu_{t-1})} - t K_{(t)}^{\epsilon(\nu_1 \dots \nu_{t-1}} \partial_\epsilon K_{(s)}^{\lambda \mu_1 \dots \mu_{s-1})} \quad (35)$$

The introduced operation for the Killing tensors is known as the *Schouten-Nijenhuis brackets*. Killing tensors form a Lie subalgebra of a Lie algebra of all totally symmetric contravariant tensor fields on the manifold with respect to these operations. The Killing tensors for which the Schouten-Nijenhuis bracket vanishes are said to be in involution.

In a simplest case when a monomial is of the first order in the momentum, the Killing tensor coincides with the Killing vector, and the *Schouten-Nijenhuis bracket* reduces to a usual commutator of two vector fields.

If there exist D non-degenerate functionally independent Killing tensors² in involution then the geodesic equations in D -dimensional spacetime are completely integrable.

Geodesic motion in the gravitational field of the most general solution describing a rotating black hole in 4 and higher dimensional spacetimes, which are asymptotically either flat or (A)dS, is a new class of physically interesting completely integrable systems.

² Some of them can be Killing vectors.

5. Killing-Yano tensors

5.1. Definitions

Killing tensors are natural symmetric generalizations of a Killing vector. Let us discuss another important generalization, known as a Killing-Yano tensor. We define first a *conformal Killing-Yano tensor* of rank p . It is an antisymmetric tensor $k_{\mu_1 \dots \mu_p}$ which obeys the following equation

$$\nabla_{(\mu_1} k_{\mu_2) \mu_3 \dots \mu_{p+1}} = g_{\mu_1 \mu_2} \tilde{k}_{\mu_3 \dots \mu_{p+1}} - (p-1) g_{[\mu_3 (\mu_1} \tilde{k}_{\mu_2) \mu_4 \dots \mu_{p+1}]} . \quad (36)$$

By tracing both sides of this equation one obtains

$$\tilde{k}_{\mu_2 \mu_3 \dots \mu_p} = \frac{1}{D-p+1} \nabla^{\mu_1} k_{\mu_1 \mu_2 \dots \mu_p} . \quad (37)$$

In the case when $\tilde{k}_{\mu_2 \dots \mu_p} = 0$ one has a *Killing-Yano* (KY) tensor. For the KY tensor \mathbf{k} the quantity

$$L_{\mu_1 \mu_2 \dots \mu_{p-1}} = k_{\mu_1 \mu_2 \dots \mu_p} p^{\mu_p} , \quad (38)$$

is parallel-propagated along the geodesic.

The ‘square’ \mathbf{K} of KY tensor \mathbf{k} , defined by the relation

$$K_{\mu\nu} = (\mathbf{k} \circ \mathbf{k})_{\mu\nu} \equiv (p-1)! k_{\mu\mu_2 \dots \mu_p} k_{\nu}^{\mu_2 \dots \mu_p} , \quad (39)$$

is a Killing tensor. Notice that not an arbitrary Killing tensor can be written in the form .

5.2. Properties of conformal Killing-Yano tensors

The (conformal) Killing-Yano tensors have the following properties:

- (i) Hodge dual of a conformal Killing-Yano tensor is a conformal Killing-Yano tensor;
- (ii) Hodge dual of a closed conformal Killing-Yano tensor is a Killing-Yano tensor;
- (iii) External product of two closed conformal Killing-Yano tensor is a closed conformal Killing-Yano tensor.

Figure 3 schematically illustrates these properties. The last of these statements was proved in [3, 4].

5.3. Principal conformal Killing-Yano tensor

Consider an antisymmetric tensor of rank 2 $h_{\mu\nu}$ which obeys the following equation

$$\nabla_{\gamma} h_{\mu\nu} = g_{\gamma\mu} \xi_{\nu} - g_{\gamma\nu} \xi_{\mu} . \quad (40)$$

This is a conformal Killing-Yano tensor. Equation (40) implies the following relations

$$\nabla_{[\gamma} h_{\mu\nu]} = 0 , \quad \xi_{\mu} = \frac{1}{D} \nabla^{\nu} h_{\nu\mu} . \quad (41)$$

The first of these relations means that \mathbf{h} is closed. Here $D = 2n + \varepsilon$ is the spacetime dimension. For even number of dimensions $\varepsilon = 0$, while for the odd number $\varepsilon = 1$. The tensor \mathbf{h} is non-degenerate if its matrix rank is $2n$. A *principal conformal Killing-Yano tensor* is a non-degenerate closed conformal Killing-Yano tensor of rank 2. The existence of the principal conformal Killing-Yano tensor for the most general known solution [5] for higher dimensional rotating black holes with spherical topology of the horizon was proved in [6, 7].

It is possible to show that the vector ξ^{μ} defined by the second equation in (41) is a Killing vector. We call it a *primary Killing vector* [8].

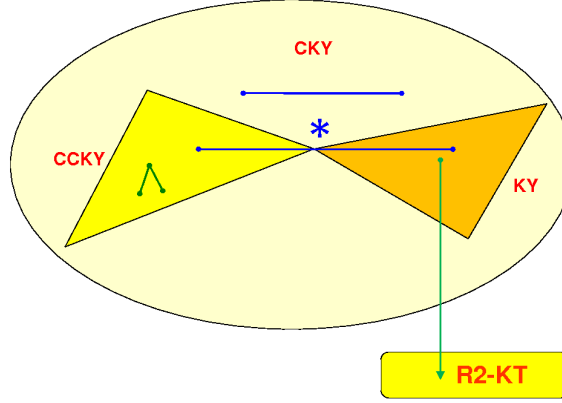


Figure 3. Schematic illustration of the properties of closed conformal Killing-Yano tensors (CCKY), Killing-Yano tensors (KY), and Killing tensors of rank 2 (R2-KT).

6. Killing-Yano ‘tower’

In a spacetime with a principal conformal Killing-Yano tensor it is possible to construct a set of Killing vectors and tensors, which we call a *Killing-Yano ‘tower’*. The idea of this construction is following. For a given principal conformal Killing-Yano tensor \mathbf{h} one can define the a set of $n - 1$ objects

$$\mathbf{h}^{\wedge j} = \underbrace{\mathbf{h} \wedge \mathbf{h} \wedge \dots \wedge \mathbf{h}}_{j \text{ times}}, \quad j = 1, \dots, n - 1. \quad (42)$$

Each of these objects is a closed conformal Killing-Yano tensor, so that their Hodge dual are Killing-Yano tensors

$$\mathbf{k}_{(j)} = *\mathbf{h}^{\wedge j}. \quad (43)$$

By taking ‘squares’ of these tensors one obtains $n - 1$ Killing tensors of the second rank

$$\mathbf{K}_{(j)} = \mathbf{k}_{(j)} \circ \mathbf{k}_{(j)}. \quad (44)$$

If ξ^ν is a primary Killing vector then it is possible to show that

$$\xi_{(j)}^\mu = K_{(j)\nu}^\mu \xi^\nu \quad (45)$$

are again Killing vectors. Thus one obtains n Killing vectors. In the odd dimensional spacetime there exists an additional Killing vector $\eta = *\mathbf{h}^{\wedge n}$. A set of constructed $n + \varepsilon$ Killing vectors, $n - 1$ Killing tensors, and one trivial Killing tensor (\mathbf{g}) gives D conserved quantities for the geodesic motion. It is possible to show that the corresponding integrals of motion are independent and in involution. Thus geodesic motion equations are completely integrable in a spacetime with a principal conformal Killing-Yano tensor [9, 10, 11].

7. Canonical form of metric

7.1. Canonical coordinates

In a spacetime with a principal conformal Killing-Yano tensor \mathbf{h} there exists a special convenient choice of coordinates. Consider the following eigen-value problem

$$h^\mu{}_\nu m_{\pm a}^\nu = \mp i x_a m_{\pm a}^\mu. \quad (46)$$

Complex eigen-vectors $m_{\pm a}^\mu$ can be written as

$$m_{\pm a}^\mu = e_a^\mu \pm i e_{\hat{a}}^\mu, \quad (47)$$

where \mathbf{e}_a and $\mathbf{e}_{\hat{a}}$ are mutually orthogonal normalised real vectors. A non-degenerate \mathbf{h} has n different eigen-values x_a ($a = 1, \dots, n$), and the corresponding eigen-space for each of these eigen-values is two dimensional (see [12]). One can use x_a as n coordinates on the spacetime manifold. We call them *Darboux coordinates*. For each of the Killing vectors ξ from the Killing-Yano ‘tower’ one can introduce a Killing parameter so that the integral line of this vector field is a solution of the equation

$$\frac{dx^\mu}{d\psi} = \xi^\mu. \quad (48)$$

This gives us $n + \varepsilon$ Killing coordinates ψ_j ($j = 0, \dots, n - \varepsilon$). Total number of Darboux and Killing coordinates is D , which is sufficient for using as coordinate system in the D dimensional spacetime manifold. We call these coordinates *canonical*.

7.2. Off-shell canonical metric

In these coordinates the metric of the spacetime is of the form [13, 15, 14]

$$ds^2 = \sum_{a=1}^n \left[\frac{U_a}{X_a} (dx_a)^2 + \frac{X_a}{U_a} \left(\sum_{j=0}^{n-1} A_a^{(j)} d\psi_j \right)^2 \right] - \frac{\varepsilon c}{A^{(n)}} \left(\sum_{j=0}^n A^{(j)} d\psi_j \right)^2. \quad (49)$$

Here

$$U_a = \prod_{b \neq a} (x_b^2 - x_a^2), \quad X_a = X_a(x_a), \quad (50)$$

$$\prod_{a=1}^n (1 + \lambda x_a^2) = \sum_{j=0}^n \lambda^j A^{(j)}, \quad (1 + \lambda x_b^2) \prod_{a=1}^n (1 + \lambda x_a^2) = \sum_{k=0}^{n-1} \lambda^k A_b^{(k)}. \quad (51)$$

A potential \mathbf{b} , which generates the principal conformal Killing-Yano tensor \mathbf{h} ,

$$\mathbf{h} = d\mathbf{b}, \quad (52)$$

in the canonical coordinates is

$$\mathbf{b} = \frac{1}{2} \sum_{k=0}^{n-1} A^{(k+1)} d\psi_k. \quad (53)$$

The metric (49) is of the algebraical type D. As we indicated above, geodesic equations in this metric are completely integrable. Besides this, it also has the following nice properties. In the metric (49):

- (i) Hamilton-Jacobi and Klein-Gordon equations allow complete separation of variables [16];
- (ii) Massive Dirac equation is separable [17];
- (iii) Stationary string equations are completely integrable [18];
- (iv) Tensorial gravitational perturbation equations are separable [19];
- (v) Equations of the parallel transport along timelike and null geodesics can be integrated [20, 21];
- (vi) Equations for charged particle motion in such a spacetime in the presence of a test electromagnetic field are completely integrable provided this field is generated by the primary Killing vector [22].

7.3. On-shell metric

The metric (49) is valid for any geometry with a principal conformal Killing-Yano tensor. We call this metric *off-shell*, since it does not obey the Einstein equations. Let us consider now on-shell geometry, that is assume that the metric obeys the equations

$$R_{\mu\nu} = (D - 1)\Lambda g_{\mu\nu}. \quad (54)$$

Λ is the cosmological constant. The Einstein equations restrict arbitrary functions $X_a(x_a)$, which enter (49), so that they take the form of polynomials [5, 23]

$$X_a = b_a x_a + \sum_{k=0}^n c_k x_a^{2k}. \quad (55)$$

As a result, the solution depends on $D - \varepsilon$ arbitrary parameters. This solution coincides with the most general solution for higher dimensional black holes in either asymptotically flat ($\Lambda = 0$), or asymptotically (A)dS spacetime, obtained in [5]. Arbitrary constants, which enter this solutions are: the cosmological constant Λ , the mass M , $(n - 1 + \varepsilon)$ rotation parameters, and $(n - 1 - \varepsilon)$ ‘NUT’ parameters. In the 4D case this is a Kerr-NUT-(A)dS metric.

7.4. Further developments

Let us mention two recent developments of the above described results.

(1) In our discussion we assumed that the closed conformal Killing-Yano tensor is non-degenerate. In particular, this implies that there exists a set of n different eigen-values of \mathbf{h} , which we used as Darboux coordinates. In a degenerate case, there may exist several eigen-values which are constants, and some of these constants can vanish. The general form of the canonical metric for such degenerate cases was constructed in [12].

(2) We discussed vacuum (with cosmological constant) solutions of the higher dimensional Einstein equations. An interesting generalization to a non-vacuum case was obtained recently in [24]. The authors consider a *five dimensional* minimally coupled gauged supergravity, which includes gravity and the Maxwell field with a Chern-Simons term. The corresponding Lagrangian density is

$$L = *(R + \Lambda) - \frac{1}{2}F \wedge *F + \frac{1}{3\sqrt{3}}F \wedge F \wedge A. \quad (56)$$

The corresponding Einstein-Maxwell equations are

$$R_{\mu\nu} + \frac{1}{3}\Lambda = \frac{1}{2}(F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{6}g_{\mu\nu}F^2), \quad (57)$$

$$dF = 0, \quad d * F - \frac{1}{\sqrt{3}}F \wedge F = 0. \quad (58)$$

The main result of this work is the following. One can modify the covariant derivative by including a non-vanishing torsion $T = \frac{1}{\sqrt{3}}*F$, and generalize the equation (40), by substituting the modified derivative instead of the covariant one. The authors demonstrated that the generalized principal conformal Killing-Yano tensor generates a ‘tower’ of integrals of motion which provides complete integrability of a charged particle motion in these spaces. An interesting example of a charged rotating black hole solution in this theory was obtained in [25]. It is interesting, that the corresponding metric is of a general algebraical type.

7.5. Acknowledgments

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